**Suggested solutions to the Contract Theory exam on Jan. 7, 2013**
**VERSION: 7 February 2013**

**Question 1 (adverse selection)**

a) **Suppose $\tau = 0$ and that P can observe A's type. Then both types will be offered a contract with full insurance (so that $\overline{u}_N = \overline{u}_A$ and $\underline{u}_N = \underline{u}_A$). Explain, in words, the economic logic behind this result.**

- The two crucial assumptions that lead to this result are that (i) A is risk averse and (ii) P is risk neutral. The objective of P is to maximize its (expected) payoff. With $\tau = 0$ there is no tax that can distort P's behavior. Moreover, under first best, the only constraints are the individual rationality constraints. Therefore, it is in the interest of P to choose A's level of insurance (for any given price A must pay for this insurance) in a way that makes A's payoff as large as possible, at least as long as this can be done at no cost for P. For if A's payoff from the insurance is higher, then P can charge more for the insurance without making A prefer his outside option. Given that A is risk averse and P is risk neutral, providing A with more insurance leads to a higher payoff for A at no cost for P. Hence the first-best optimum involves P providing full insurance to A and then choosing the effective price for this insurance so high that each type of A is indifferent between the outside option and the insurance contract.
  - The reason why the logic above does not apply under second best is that then P has a smaller number of instruments available: P cannot observe A's type, which means that the level of A's insurance must also be such that A voluntarily chooses the right contract.

b) **Prove that IC-low and IC-high jointly imply the following:**

- **If the high-demand type is underinsured ($\overline{u}_N > \overline{u}_A$), so is the low-demand type ($\underline{u}_N > \underline{u}_A$).**
- Add up the ICs:

$$\left(1-\overline{\theta}\right)\overline{u}_N + \overline{\theta}\,\overline{u}_A + \left(1-\underline{\theta}\right)\underline{u}_N + \underline{\theta}\,\underline{u}_A \geq \left(1-\overline{\theta}\right)\underline{u}_N + \overline{\theta}\,\underline{u}_A + \left(1-\underline{\theta}\right)\overline{u}_N + \underline{\theta}\,\overline{u}_A.$$

Re-arranging and noticing that some terms cancel out, we obtain

$$-\left(\overline{\theta}-\underline{\theta}\right)\overline{u}_N + \left(\overline{\theta}-\underline{\theta}\right)\overline{u}_A + \left(\overline{\theta}-\underline{\theta}\right)\underline{u}_N - \left(\overline{\theta}-\underline{\theta}\right)\underline{u}_A \geq 0.$$

Since $\overline{\theta} > \underline{\theta}$, the inequality simplifies to

$$-\overline{u}_N + \overline{u}_A + \underline{u}_N - \underline{u}_A \geq 0$$

or

$$\underline{u}_N - \underline{u}_A \geq \overline{u}_N - \overline{u}_A.$$

The last inequality implies that if $\overline{u}_N > \overline{u}_A$, so that the right-hand side is positive, then also the left-hand side must be positive, meaning $\underline{u}_N > \underline{u}_A$. That is, the ICs jointly imply that if the high-demand type is underinsured, then so is the low-demand type, which is what we were asked to show.

**c) Argue, in words, that the second-order condition to P's problem is satisfied.**

- Given the rewritten formulation of P's problem, it is clear from inspection that the objective function is strictly concave. We can see this because we know that the function $h$ is strictly convex and it enters the objective with a negative sign. Moreover, there are no cross terms; that is, all the cross derivatives of the objective, e.g. first w.r.t. $\underline{u}_N$ and then w.r.t $\underline{u}_A$, are zero. Those observations are sufficient to be able to conclude that the objective is strictly concave.
- The constraints are all linear in the choice variables. This means that, almost everywhere, the border of the feasible set (i.e., the set of utility levels for which all constraints are satisfied) is linear, which suggests that there should be a good chance that the feasible set does not create any problems for the second-order condition.
  - However, this reasoning does not guarantee that the feasible set really is convex. At a point where two constraints meet, the corner that is created could point "in the wrong direction". Therefore, to rule out this possibility, one would have to check the second-order condition directly.

**d) Assume that the constraints (IR-high) and (IC-low) are lax at the second-best optimum (so that they can be disregarded). Solve P's problem and characterize the optimal second-best utility levels, $\overline{u}_N^{SB}, \overline{u}_A^{SB}, \underline{u}_N^{SB}, \underline{u}_A^{SB}$. Will the low- and high-demand type, respectively, be underinsured, fully insured or overinsured at the second-best optimum?**

- First assume, as the question lets us do, that IC-low and IR-high does not bind at the optimum.
- The remaining problem thus has two constraints. We solve this by setting up a Lagrangian and then take first-order conditions w.r.t. the four choice variables. The Lagrangian is:

$$\begin{aligned}
\mathcal{L} \quad = \quad & \frac{\upsilon}{1+\tau}\left[w - \underline{\theta}(1+\tau)d - [1-\underline{\theta}(1+\tau)]\,h(\underline{u}_N) - \underline{\theta}(1+\tau)\,h(\underline{u}_A)\right] \\
& + \frac{1-\upsilon}{1+\tau}\left[w - \overline{\theta}(1+\tau)d - \left[1-\overline{\theta}(1+\tau)\right]h(\overline{u}_N) - \overline{\theta}(1+\tau)\,h(\overline{u}_A)\right] \\
& + \lambda\left[(1-\underline{\theta})\,\underline{u}_N + \underline{\theta}\underline{u}_A - \underline{U}^*\right] \\
& + \mu\left[\left(1-\overline{\theta}\right)\overline{u}_N + \overline{\theta}\overline{u}_A - \left(1-\overline{\theta}\right)\underline{u}_N - \overline{\theta}\underline{u}_A\right],
\end{aligned}$$

where $\lambda$ is the shadow price associated with IR-low and $\mu$ is the shadow price associated with IC-high.

- FOC w.r.t. $\overline{u}_N$:

$$\frac{\partial \mathcal{L}}{\partial \overline{u}_N} = -\frac{1-\upsilon}{1+\tau}\left[1-\overline{\theta}\left(1+\tau\right)\right] h'\left(\overline{u}_N\right) + \mu\left(1-\overline{\theta}\right) = 0$$

or

$$\boxed{\tfrac{1-\upsilon}{1+\tau}\left[1-\overline{\theta}\left(1+\tau\right)\right] h'\left(\overline{u}_N\right) = \mu\left(1-\overline{\theta}\right).} \tag{1}$$

  - This implies that $\boxed{\mu > 0}$; i.e., $\boxed{\text{IC-high binds at the optimum}}$.

- FOC w.r.t. $\underline{u}_N$:

$$\frac{\partial \mathcal{L}}{\partial \underline{u}_N} = -\frac{\upsilon}{1+\tau}\left[1-\underline{\theta}\left(1+\tau\right)\right] h'\left(\underline{u}_N\right) + \lambda\left(1-\underline{\theta}\right) - \mu\left(1-\overline{\theta}\right) = 0$$

or

$$\boxed{\tfrac{\upsilon}{1+\tau}\left[1-\underline{\theta}\left(1+\tau\right)\right] h'\left(\underline{u}_N\right) = \lambda\left(1-\underline{\theta}\right) - \mu\left(1-\overline{\theta}\right).} \tag{2}$$

  - This implies that $\boxed{\lambda > 0}$; i.e., $\boxed{\text{IR-low binds at the optimum}}$.

- FOC w.r.t. $\overline{u}_A$:

$$\frac{\partial \mathcal{L}}{\partial \overline{u}_A} = -\frac{1-\upsilon}{1+\tau}\overline{\theta}\left(1+\tau\right) h'\left(\overline{u}_A\right) + \mu\overline{\theta} = 0$$

or

$$\boxed{\left(1-\upsilon\right) h'\left(\overline{u}_A\right) = \mu.} \tag{3}$$

- FOC w.r.t. $\underline{u}_A$:

$$\frac{\partial \mathcal{L}}{\partial \underline{u}_A} = -\frac{\upsilon}{1+\tau}\left[\underline{\theta}\left(1+\tau\right) h'\left(\underline{u}_A\right)\right] + \lambda\underline{\theta} - \mu\overline{\theta} = 0$$

or

$$\boxed{\upsilon\underline{\theta} h'\left(\underline{u}_A\right) = \lambda\underline{\theta} - \mu\overline{\theta}.} \tag{4}$$

- The solution to the problem is thus characterized by the four first-order conditions (1)-(4) and the two binding constraints, IR-low and IC-high. In order to answer the remaining questions under d), we manipulate the FOCs a little bit further.

- Plug (3) into (1):

$$\begin{aligned}\frac{1-\upsilon}{1+\tau}\left[1-\overline{\theta}\left(1+\tau\right)\right] h'\left(\overline{u}_N\right) &= \mu\left(1-\overline{\theta}\right) \\ &= \underbrace{\left(1-\upsilon\right) h'\left(\overline{u}_A\right)}_{=\mu}\left(1-\overline{\theta}\right)\end{aligned}$$

or

$$\frac{h'\left(\overline{u}_N\right)}{h'\left(\overline{u}_A\right)} = \frac{\left(1-\overline{\theta}\right)\left(1+\tau\right)}{1-\overline{\theta}\left(1+\tau\right)} = 1 + \frac{\tau}{1-\overline{\theta}\left(1+\tau\right)} > 1$$

or

$$h'\left(\overline{u}_N\right) > h'\left(\overline{u}_A\right) \Leftrightarrow \overline{u}_N > \overline{u}_A.$$

  - That is, there is underinsurance for the high-demand type.

- Multiply (2) by $\underline{\theta}$:

$$\frac{\underline{\theta}\upsilon}{1+\tau}\left[1-\underline{\theta}\left(1-\sigma\right)\right]h'\left(\underline{u}_N\right)=\lambda\underline{\theta}\left(1-\underline{\theta}\right)-\mu\underline{\theta}\left(1-\overline{\theta}\right). \tag{5}$$

- Multiply (4) by $(1-\underline{\theta})$:

$$\upsilon\underline{\theta}\left(1-\underline{\theta}\right)h'\left(\underline{u}_A\right)=\lambda\underline{\theta}\left(1-\underline{\theta}\right)-\mu\overline{\theta}\left(1-\underline{\theta}\right). \tag{6}$$

- Subtract (6) from (5):

$$\begin{aligned}&\frac{\underline{\theta}\upsilon}{1+\tau}\left[1-\underline{\theta}\left(1+\tau\right)\right]h'\left(\underline{u}_N\right)-\upsilon\underline{\theta}\left(1-\underline{\theta}\right)h'\left(\underline{u}_A\right)\\=\;&\left[\lambda\underline{\theta}\left(1-\underline{\theta}\right)-\mu\underline{\theta}\left(1-\overline{\theta}\right)\right]-\left[\lambda\underline{\theta}\left(1-\underline{\theta}\right)-\mu\overline{\theta}\left(1-\underline{\theta}\right)\right]\end{aligned}$$

or

$$\begin{aligned}&\frac{\underline{\theta}\upsilon}{1+\tau}\left\{\left[1-\underline{\theta}\left(1+\tau\right)\right]h'\left(\underline{u}_N\right)-\left(1-\underline{\theta}\right)\left(1+\tau\right)h'\left(\underline{u}_A\right)\right\}\\=\;&\mu\left[\overline{\theta}\left(1-\underline{\theta}\right)-\underline{\theta}\left(1-\overline{\theta}\right)\right]=\mu\left(\overline{\theta}-\underline{\theta}\right)\end{aligned}$$

or

$$\begin{aligned}&\frac{\underline{\theta}\upsilon}{1+\tau}\left\{\left[1-\underline{\theta}\left(1+\tau\right)\right]h'\left(\underline{u}_N\right)-\left[1+\tau-\underline{\theta}\left(1+\tau\right)\right]h'\left(\underline{u}_A\right)\right\}\\=\;&\mu\left(\overline{\theta}-\underline{\theta}\right)\end{aligned}$$

or

$$\frac{\underline{\theta}\upsilon}{1+\tau}\left\{\left[1-\underline{\theta}\left(1+\tau\right)\right]\left[h'\left(\underline{u}_N\right)-h'\left(\underline{u}_A\right)\right]-\tau h'\left(\underline{u}_A\right)\right\}=\mu\left(\overline{\theta}-\underline{\theta}\right)$$

or

$$\frac{\underline{\theta}\upsilon}{1+\tau}\left[1-\underline{\theta}\left(1+\tau\right)\right]\left[h'\left(\underline{u}_N\right)-h'\left(\underline{u}_A\right)\right]=\mu\left(\overline{\theta}-\underline{\theta}\right)+\frac{\underline{\theta}\upsilon\tau h'\left(\underline{u}_A\right)}{1+\tau}.$$

  - We have shown above that $\mu > 0$. Also, by assumption $\tau > 0$. Therefore the right-hand side of the last expression is positive. Hence also the the left-hand side must be positive, which means that $h'(\underline{u}_N) > h'(\underline{u}_A)$ or (since $h'' > 0$) $\underline{u}_N > \underline{u}_A$. That is, there is underinsurance also for the low-demand type type.

**Question 2 (moral hazard)**

**Consider the following moral hazard model with mean-variance preferences that we studied in the course. There is one (single) agent, A, and one principal, P. A chooses an effort level $e \in \Re_+$, thereby incurring the cost $c(e) = \frac{1}{2}e^2$. Given a choice of $e$, the output (i.e., A's performance) equals $q = e + z$, where $z$ is an exogenous random term drawn from a normal distribution with mean zero and variance $\nu$. It is assumed that P can observe $q$ but not $e$. Moreover, neither P nor A can observe $z$. A's wage (i.e., the transfer from P to A) can only be contingent on the output $q$. It is restricted to be linear in $q$:**

$$t = \alpha + \beta q = \alpha + \beta(e + z).$$

**A is risk averse and has a CARA utility function: $U = -\exp[-r(t - c(e))]$, where $r$ $(> 0)$ is the coefficient of absolute risk aversion. Therefore A's expected utility is**

$$EU = -\int_{-\infty}^{\infty} \exp[-r(t - c(e))] f(z)\, dz,$$

**where $f(z)$ is the density of the normal distribution. P's objective function is**

$$V = q - t = q - \alpha - \beta q = (1 - \beta)(e + z) - \alpha,$$

**which in expected terms becomes $EV = (1 - \beta)e - \alpha$. It is also assumed that A's outside option utility is $\widehat{U} = -\exp\left[-r\widehat{t}\right]$, where $\widehat{t} > 0$. The timing of events is as follows.**

1. **P chooses the contract parameters, $\alpha$ and $\beta$.**

2. **A accepts or rejects the contract and, if accepting, chooses an effort level.**

3. **The noise term $z$ is realized and A and P get their payoffs.**

**Answer the following questions:**

a) **Solve for the $\beta$-parameter in the second-best optimal contract, denoted $\beta^{SB}$ (you do not need to solve for $\alpha^{SB}$, and you will not get any credit if you nevertheless do that). You should make use of the following (well-known) result:**

$$EU = -\exp\left[-r\left(\alpha + \beta e - \frac{1}{2}e^2 - \frac{1}{2}\nu r \beta^2\right)\right].$$

- P's chooses the parameters in the contract, $\alpha$ and $\beta$. In addition, P can effectively choose A's effort $e$, because P designs the incentives that A faces when deciding what effort to make. We can thus think of P as choosing $\alpha$, $\beta$, and $e$ in order to maximize his expected payoff, subject to A's incentive compatibility constraint. In addition, A's individual rationality constraint must be satisfied. P's problem:

$$\max_{\alpha,\beta,e} \left\{ \overbrace{(1-\beta)\,e-\alpha}^{=EV} \right\}$$

subject to

$$\overbrace{-\int_{-\infty}^{\infty} \exp\left[-r\left(t-c\left(e\right)\right)\right] f\left(z\right) dz}^{=EU} \geq -\exp\left[-r\widehat{t}\right], \qquad \text{(IR)}$$

$$e \in \arg\max_{e'} EU\left(e'\right). \qquad \text{(IC)}$$

The IC constraint says that $e$ indeed maximizes A's utility among all the $e$'s that A could choose. The IR constraint says that A's expected utility if accepting the contract is at least as large as his utility from his outside option; this therefore ensures that A wants to participate.

- The IC constraint above is actually a whole set of infinitely many constraints. In order to reduce these to one single IC constraint, we can make use of the first-order approach, which means that we replace IC above with the first-order condition from A's maximization problem (for some arbitrary values of the contract parameters $\alpha$ and $\beta$). From the question we have that A's expected utility can be written as

$$EU = -\exp\left[-r\left(\alpha+\beta e-\frac{1}{2}e^2-\frac{1}{2}\nu r\beta^2\right)\right].$$

Maximizing $EU$ is equivalent to maximizing a monotone transformation of this expression, so we can without loss of generality let A maximize

$$\widetilde{EU} = \alpha+\beta e-\frac{1}{2}e^2-\frac{1}{2}\nu r\beta^2. \qquad (7)$$

- We have

$$\frac{\partial \widetilde{EU}}{\partial e} = \beta - e = 0$$

Therefore A's optimal effort level is

$$e = \beta. \qquad (8)$$

- We can write the IR constraint as

$$\begin{aligned}
-\int_{-\infty}^{\infty} \exp\left[-r\left(t-c\left(e\right)\right)\right] f\left(z\right) dz &\geq -\exp\left[-r\widehat{t}\right] \Leftrightarrow \\
-\exp\left[-r\left(\alpha+\beta e-\frac{1}{2}e^2-\frac{1}{2}\nu r\beta^2\right)\right] &\geq -\exp\left[-r\widehat{t}\right] \Leftrightarrow \\
\exp\left[-r\left(\alpha+\beta e-\frac{1}{2}e^2-\frac{1}{2}\nu r\beta^2\right)\right] &\leq \exp\left[-r\widehat{t}\right] \Leftrightarrow \\
-r\left(\alpha+\beta e-\frac{1}{2}e^2-\frac{1}{2}\nu r\beta^2\right) &\leq -r\widehat{t} \Leftrightarrow \\
\alpha+\beta e-\frac{1}{2}e^2-\frac{1}{2}\nu r\beta^2 &\geq \widehat{t} \Leftrightarrow \\
\alpha &\geq \widehat{t}-\beta e+\frac{1}{2}e^2+\frac{1}{2}\nu r\beta^2
\end{aligned}$$

  Plugging in (8) in this inequality, we obtain

$$\begin{aligned}
\alpha &\geq \widehat{t}-\beta^2+\frac{1}{2}\beta^2+\frac{1}{2}\nu r\beta^2 \\
&= \widehat{t}-\frac{1}{2}\left(1-\nu r\right)\beta^2.
\end{aligned}$$

  Plugging in (8) into P's objective function $EV=\left(1-\beta\right)e-\alpha$ , we have

$$EV=\left(1-\beta\right)\beta-\alpha.$$

- Using the above results, P's problem becomes

$$\max_{\alpha,\beta}\left\{\left(1-\beta\right)\beta-\alpha\right\} \qquad \text{subject to}$$

$$\alpha \geq \widehat{t}-\frac{1}{2}\left(1-\nu r\right)\beta^2. \tag{IR}$$

- It is clear that IR must bind, as the objective is decreasing in $\alpha$ and the constraint is tightened as $\alpha$ is lowered (thus P wants to lower $\alpha$ until the constraint says stop). We thus have $\alpha=\widehat{t}-\frac{1}{2}\left(1-\nu r\right)\beta^2$. Plugging this value of $\alpha$ into the objective yields the following unconstrained problem:

$$\boxed{\max_\beta\left\{\beta-\tfrac{1}{2}\left(1+\nu r\right)\beta^2-\widehat{t}\right\},}$$

  with the first-order condition

$$1-\left(1+\nu r\right)\beta=0 \Rightarrow \beta^{SB}=\frac{1}{1+\nu r}.$$

b) **[You are encouraged to attempt parts b), c) and d) even if you have not been able to answer parts a).] Does the agent get any rents at the second-best optimum? Do not only answer yes or no, but also explain how you can tell. [PLEASE TURN THE PAGE!]**

- No, he does not get any rents at the second-best optimum. "Rents" are defined as any payoff from accepting the contract that exceeds the outside option payoff. However, we saw under a) that the IR constraint binds at the optimum, which means that A does not get any rents.

c) **The first-best values of the effort level and the $\beta$-parameter equal $e^{FB} = 1$ and $\beta^{FB} = 0$, respectively. How do these values relate to the corresponding second-best values? In particular, is there under- or overprovision of effort at the second-best optimum?**

- We have from the above analysis that $\beta^{SB} = e^{SB} = \frac{1}{1+\nu r}$. We see that there is underprovision of effort (as $e^{SB} < e^{FB}$). We also see that the beta-parameter is too high relative to the first best level ($\beta^{SB} > \beta^{FB}$).

d) **Consider the limit case where $r \to 0$. Explain what happens to the relationship between the second-best and the first-best effort levels. Also explain the intuition for this result.**

- In the limit where $r \to 0$, A is risk neutral. We see from above that in that limit, $e^{SB} = 1$. That is, the second-best effort level coincides with the first-best level: there is no inefficiency in spite of the fact that there asymmetric information. The reason why this can occur is that when risk neutral, A doesn't mind bearing risk. Therefore P can incentivize A very strongly, so that indeed $\beta^{SB} \to 1$ as $r \to 0$: A's compensation depends fully on the stochastic variation, so he makes the same decision as P would have made if he had been in A's job.
- The intuition is the same as we have discussed in other parts of the course, for example in the 2x2 moral hazard model with a risk neutral agent who is not protected by limited liability. There we explained the intuition as follows:
  - The economic meaning of the fact that A is risk neutral is that he cares only about whether his payment $t$ is large enough *on average*. Hence, P can, without violating the participation constraint, incentivize A by giving him a negative payment (in practice a penalty) in case of a low output. More generally, P can achieve the first-best outcome by making A the residual claimant:
    * Then A effectively buys the right to receive any returns: "the firm is sold to the agent".
    * Thereby, the effort level is chosen by the same individual who bears the consequences of the choice.
    * In this situation A makes the same effort choice as P would have made.